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2001 J. Phys. A: Math. Gen. 34 L169

(http://iopscience.iop.org/0305-4470/34/13/103)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) L169–L178 www.iop.org/Journals/ja PII: S0305-4470(01)21358-4

LETTER TO THE EDITOR

Gauge theory of Riemann ellipsoids

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Received 24 January 2001, in final form 16 February 2001

Abstract

The classical theory of Riemann ellipsoids is formulated naturally as a gauge theory based on a principal G-bundle $\mathcal{P}$. The structure group $G = SO(3)$ is the vorticity group, and the bundle $\mathcal{P} = GL_{+}(3, \mathbb{R})$ is the connected component of the general linear group. The base manifold is the space of positive-definite real 3×3 symmetric matrices, identified geometrically with the space of inertia ellipsoids. The angular momentum is not the only conserved quantity. The Kelvin circulation is also conserved as a consequence of gauge invariance. The bundle $\mathcal{P}$ is a Riemannian manifold whose metric is determined by the kinetic energy. Nonholonomic constraints determine connexions on the bundle. In particular, the trivial connexion corresponds to rigid body motion, the natural Riemannian connexion to irrotational flow, and the invariant connexion to the falling cat.

PACS numbers: 4550P, 9510C, 2160F, 0240

1. Introduction

This Letter formulates the classical theory of Riemann ellipsoids as a gauge theory. In the usual gauge method for isolated mechanical systems, the structure group is the rotation group, and the gauge conserved quantity is the angular momentum [1–6]. The gauge theory of Riemann ellipsoids is a fundamentally different model with the vorticity group as the structure group and the Kelvin circulation as a gauge-conserved quantity in addition to the angular momentum.

As defined by Dirichlet [7] and Riemann [8] in 1860, a Riemann ellipsoid is a self-gravitating, constant mass-density fluid with an ellipsoidal boundary and with a velocity field that is a linear function of the Cartesian position coordinates in an inertial centre-of-mass frame. The classical theory of the equilibrium and stability of rotating Riemann ellipsoids was clarified by Lebovitz [9] and Chandrasekhar [10] a century later and applied to the description of astrophysical systems [11–15] and gaseous plasmas [16].

With minor modifications, Riemann ellipsoid theory may be applied to fluids whose density is not uniform and to discrete systems of particles. Rapidly rotating atomic nuclei may be modelled as Riemann ellipsoids when the gravitational self-energy is replaced by the sum of the repulsive Coulomb self-energy among the protons and an attractive surface energy that approximates the strong interactions among the nucleons [17, 18]. There is a quantum mechanical mean field theory of Riemann ellipsoids [19–21]. The equations of motion form a Hamiltonian dynamical system [22] and a finite-dimensional Lax system [23].

Because the velocity field is assumed linear, the Riemann ellipsoid configuration space is the connected component of the general linear group

$$\mathcal{P} = GL_+(3, \mathbb{R}) = \{\xi \in M_3(\mathbb{R}) \mid \det \xi > 0\}. \tag{1}$$

The space of all inertia ellipsoids is identified with the manifold Q of all positive-definite real symmetric matrices. The relationship between $\mathcal{P}$ and Q is given by the surjective mapping π:

$$\begin{aligned} &\mathcal{P} \xrightarrow{\pi} Q \\ &\xi \longmapsto q = \xi\xi^t. \end{aligned} \tag{2}$$

The group $\mathcal{P}$ acts on itself by multiplication on the left or on the right. These are two different geometrical transformations with distinct physical interpretations. For example, left multiplication of elements $\xi \in \mathcal{P}$ by elements r of the subgroup $SO(3)$, $\xi \longmapsto L_r\xi = r\xi$ corresponds to physical rotations. In contrast, right multiplication $\xi \longmapsto R_g\xi = \xi g^{-1}$ by $g \in G = SO(3)$ corresponds to vortex motion. The difference in interpretation originates in the distinct induced group actions on the ellipsoidal space. With respect to left multiplication by the rotation group element $r \in SO(3)$, an ellipsoid with inertia tensor $q = \pi(\xi)$ is transformed into a rotated ellipsoid with inertia tensor $\pi(L_r\xi) = r\xi\xi^t r^t = rqr^t$. But right multiplication by the vorticity group element $g \in G$ leaves the inertia ellipsoid invariant, $\pi(R_g\xi) = \xi g^{-1} g \xi^t = q$, since $g^{-1} = g^t$, or

$$\pi \circ R_g = \pi \qquad \text{for all } g \in G. \tag{3}$$

The projection π is right invariant with respect to the group G. Hence, the configuration space $\mathcal{P}$ is a principal fibre bundle over the base manifold Q with structure group G [24, 25].

This letter shows that the classical theory of Riemann ellipsoids is expressed naturally in terms of the differential geometry of the bundle $\mathcal{P}$. A connexion, or differential geometric structure, on the bundle $\mathcal{P}$ is physically equivalent to a nonholonomic constraint on the vortex velocity field. The nonholonomic constraints to irrotational flow and the 'falling cat' problem correspond to the Riemannian connexion and the invariant connexion, respectively. Littlejohn and Reinsch [26] reviewed the relationship between gauge theory and traditional physics approaches to nonholonomic constraints, especially in atomic and molecular science, while Massa and Pagani [27] and Bates and Sniatycki [28] provide mathematical overviews of the nonholonomic problem.

2. Kinematics

The kinematics of Riemann ellipsoids in the gauge formalism is obtained by certain local trivializations of the bundle $\mathcal{P}$, which separate the degrees of freedom into rotational, vibrational and vortex components. Every group element $\xi \in \mathcal{P}$ can be expressed as a product of three matrices, $\xi = R^t A\, S$, where R, S are real orthogonal matrices and A is a diagonal matrix with real positive entries in descending order. The projection $q = \pi(\xi) = R^t A^2 R$ in the ellipsoidal space of a bundle point ξ shows that the entries of the square of A are the eigenvalues of q and R is an orthogonal matrix that diagonalizes q. Physically R rotates the body into the principal axis frame, and the entries of A are the lengths of the inertia ellipsoid's principal axes. Because eigenvalues are unique, the diagonal matrix A is determined uniquely by q. The eigenspaces are also uniquely defined by q. If the eigenvalues are distinct, the eigenspaces are one dimensional and each row of R is unique up to a sign. Thus, when restricted to suitable open neighbourhoods, the matrices R and A provide a local coordinate chart for the ellipsoidal space Q. Once R and A are determined by the local chart for the base manifold Q, the orthogonal matrix S in the structure group is given uniquely. A unique decomposition $\xi = R^t A\, S$, or $\xi = (q; S)$ for $q = R^t A^2 R$ and $S = A^{-1} R\xi$, in an open neighbourhood of $\mathcal{P}$ is a local trivialization of the bundle $\mathcal{P}$. The bundle $\mathcal{P}$ is only locally diffeomorphic to the Cartesian product of the base manifold Q and the structure group G.

With respect to left multiplication by elements r in the rotation group, the bundle point $\xi = R^t A\, S$ is transformed to $L_r \xi = (R\, r^t)^t A\, S$, or a rotation r is equivalent to right multiplication of the elements R of the subgroup $SO(3)$. With respect to right multiplication by elements g in the structure group, the bundle point $\xi = R^t A\, S$ is transformed to $R_g \xi = R^t A(Sg^{-1})$, or a gauge transformation g is equivalent to right multiplication of the elements S of the subgroup G.

2.1. Tangent space

Consider a curve $t \longmapsto \xi(t)$ in the bundle $\mathcal{P}$. Such a curve may be identified with the collective motion of a many-body system for which the trajectory of each particle α is constrained by $\boldsymbol{x}_\alpha(t) = \xi(t)\boldsymbol{y}_\alpha$, where $\boldsymbol{y}_\alpha$ is independent of time. The reference particle distribution $\boldsymbol{y}_\alpha$ is chosen so that its dimensionless inertia tensor is the identity matrix. With this choice the instantaneous inertia tensor of the many-body system simplifies to $q(t) = \xi\xi^t$.

The velocity vector for each particle is $\boldsymbol{v}_\alpha = \dot{\xi}\, \boldsymbol{y}_\alpha = u\, \boldsymbol{x}_\alpha$ for $u = \dot{\xi}\xi^{-1}$ and $\dot{\xi} = \mathrm{d}\xi/\mathrm{d}t$. Note that $\boldsymbol{v}_\alpha$ is a linear function of its position vector $\boldsymbol{x}_\alpha$. The velocity vector may be expressed as the value of a right-invariant vector field on the group $\mathcal{P}$ at the point ξ,

$$V(t) = \sum_{ij} (\dot{\xi}\xi^{-1} \cdot \xi)_{ij} \frac{\partial}{\partial \xi_{ij}} = -(\mathcal{R}_u)_\xi. \tag{4}$$

With respect to a local trivialization, the curve is $t \longmapsto R(t)^t A(t)\, S(t)$. At each time t, define the antisymmetric matrix $\Omega(t) = \dot{R}R^t$ in the Lie algebra $\mathfrak{so}(3)$ of the rotation group and the antisymmetric matrix $\Lambda(t) = \dot{S}S^t$ in the Lie algebra $\mathfrak{g}$ of the structure group. A basis for the space of 3×3 antisymmetric matrices is given by e_i for $i = 1, 2, 3$, where $(e_i)_{jk} \equiv \varepsilon_{ijk}$. The matrix Ω determines the angular velocity vector $\boldsymbol{\omega}$, and Λ—the vortex velocity vector $\boldsymbol{\lambda}$:

$$\Omega = \sum_i \omega_i e_i \qquad \Lambda = \sum_i \lambda_i e_i. \tag{5}$$

For such a local trivialization, the velocity of the curve in the bundle can be shown to be a sum of rotational, vibrational and vortex terms,

$$\begin{aligned} V(t) &= -(\mathcal{R}_\Omega)_R + \sum_i \dot{a}_i \left(\frac{\partial}{\partial a_i}\right)_A - (\mathcal{R}_\Lambda)_S \\ &= \sum_i \left(-\omega_i \left(\mathcal{R}_{e_i}\right)_R + \dot{a}_i \left(\frac{\partial}{\partial a_i}\right)_A - \lambda_i \left(\mathcal{R}_{e_i}\right)_S \right). \end{aligned} \tag{6}$$

The velocity vector may be expressed alternatively as a sum of right-invariant vector fields on the bundle $\mathcal{P}$ by using the identities,

$$\begin{aligned} (\mathcal{R}_{R^t \Omega R})_\xi &= -(\mathcal{R}_\Omega)_R = (\Omega R)_{ij} \left(\frac{\partial}{\partial R_{ij}}\right)_R \\ (\mathcal{R}_{R^t A^{-1} \dot{A} R})_\xi &= -\dot{a}_i \left(\frac{\partial}{\partial a_i}\right)_A \\ (\mathcal{R}_{R^t A \Lambda A^{-1} R})_\xi &= (\mathcal{R}_\Lambda)_S = -(\Lambda S)_{ij} \left(\frac{\partial}{\partial S_{ij}}\right)_S \end{aligned} \tag{7}$$

when $\xi = R^t AS$. Here $(\mathcal{R}_\Omega)_R$ denotes a right-invariant vector field on $SO(3)$ and $(\mathcal{R}_\Lambda)_S$ denotes a right-invariant vector field on G.

2.2. Riemannian structure

For X, Y in $M_3(\mathbb{R})$, define the metric at the point $\xi \in \mathcal{P}$ by

$$\boldsymbol{g}_\xi((\mathcal{R}_X)_\xi, (\mathcal{R}_Y)_\xi) = \mathrm{tr}(X(\xi\xi^t)Y^t). \tag{8}$$

This is a positive-definite bilinear form defined on each tangent space of $\mathcal{P}$ so that the bundle is a Riemannian manifold. The kinetic energy is proportional to the squared length of the velocity

$$K = (\mathcal{I}/8)\, \boldsymbol{g}_\xi(V(t), V(t)) \tag{9}$$

where $\mathcal{I}$ is a constant with the units of the moment of inertia. Expanding the velocity into the three types of motion, equation (6), the kinetic energy becomes

$$K = (\mathcal{I}/4)\big(- \mathrm{tr}(A^2\Omega^2) + \mathrm{tr}(\dot{A}^2) - \mathrm{tr}(A^2\Lambda^2) + 2\ \mathrm{tr}(\Omega A \Lambda A)\big). \tag{10}$$

The last term is due to Coriolis coupling between the rotational and vortex degrees of freedom. The derivatives of the kinetic energy with respect to the angular velocity and vortex velocity are the vectors of angular momentum and circulation, respectively,

$$L_k = \frac{\partial K}{\partial \omega_k} = (\mathcal{I}/2)\, [(a_i^2 + a_j^2)\omega_k - 2a_i a_j \lambda_k] \tag{11}$$

$$C_k = -\frac{\partial K}{\partial \lambda_k} = (\mathcal{I}/2)\, [2a_i a_j \omega_k - (a_i^2 + a_j^2)\lambda_k] \tag{12}$$

where i, j, k are cyclic.

The equations of motion are found using the Lagrangian formalism [18, 29]. Suppose that the potential energy $V(A)$ is a smooth function of the principal axes lengths. Then the potential is left- and right-invariant with respect to the rotation group and the structure group, respectively. But the metric, and hence the kinetic energy, is also left- and right-invariant with respect to the rotation and structure groups. Since the Lagrangian is the difference between the kinetic and potential energies, the two invariances, according to Noether's theorem, imply conservation laws. These are the angular momentum and Kelvin circulation. In the rotating body-fixed frame, the angular momentum and Kelvin circulation vectors precess:

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = -\boldsymbol{\omega} \times \boldsymbol{L} \qquad \text{and} \qquad \frac{\mathrm{d}\boldsymbol{C}}{\mathrm{d}t} = -\boldsymbol{\lambda} \times \boldsymbol{C}. \tag{13}$$

The two vector conservation laws in the inertial centre-of-mass frame are

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(R^t\, \boldsymbol{L}\big) = 0 \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\big(S^t\, \boldsymbol{C}\big) = 0. \tag{14}$$

3. Connexions on the Riemann ellipsoid bundle

For many mechanical systems, there are constraint forces in addition to those described by the potential energy $V(A)$. The simplest case is the rigid body for which the vortex velocity vanishes, $\boldsymbol{\lambda} = 0$. This is a holonomic constraint which reduces the configuration space to $\mathcal{Q} \cong \mathcal{P}/G$. But constraints are not typically holonomic. For example, an irrotational fluid (like a water droplet) has zero circulation, $\boldsymbol{C} = 0$. Another example is the 'falling cat' [30,31], for which the angular momentum vanishes, $\boldsymbol{L} = 0$. In these cases the vortex velocity is proportional to the angular velocity [10]

$$\frac{\lambda_k}{\omega_k} = \begin{cases} \dfrac{2a_i a_j}{a_i^2 + a_j^2} & \text{irrotational flow} \\[2ex] \dfrac{a_i^2 + a_j^2}{2a_i a_j} & \text{falling cat} \end{cases} \tag{15}$$

where i, j, k are cyclic. A nonholonomic constraint for a Riemann ellipsoid is a proportionality between the vortex and angular velocity components, $\lambda_k = f_k(A)\omega_k$ with a factor $f_k(A)$ that depends on the axes lengths. This proportionality is equivalent to a connexion on the bundle $\mathcal{P}$, as it will be shown next.

For each point ξ in the bundle, denote the tangent space by $T_\xi\mathcal{P}$. By definition, the vertical space V_ξ is the subspace of $T_\xi\mathcal{P}$ consisting of the tangents to curves in the fibre G,

$$V_\xi = \{X \in T_\xi\mathcal{P} \mid \pi_* X = 0\}. \tag{16}$$

If $\Lambda \in \mathfrak{g}$ is a Lie algebra element, then the fundamental vector field, denoted by Λ^*, is the left invariant vector field on the fibre G. A basis for V_ξ is the set of fundamental vector fields $\{e_a^*, a = 1, 2, 3\}$, where $(e_a)_{bc} = \varepsilon_{abc}$.

A connexion [25] is a smooth assignment of a horizontal subspace H_ξ of the tangent space $T_\xi\mathcal{P}$ to each point $\xi \in \mathcal{P}$ such that

$$(1) \quad T_\xi\mathcal{P} = H_\xi \oplus V_\xi \tag{17}$$

$$(2) \quad H_{\xi\cdot g} = (R_g)_* H_\xi. \tag{18}$$

Because the kernel of π_* at $\xi \in \mathcal{P}$ is the vertical subspace V_ξ, its image is $T_q Q$, and the tangent space $T_\xi\mathcal{P}$ is a direct sum of vertical and horizontal subspaces, the linear transformation π_* is an isomorphism from the horizontal subspace onto the tangent space of the base manifold $\pi_* : H_\xi \longrightarrow T_q Q$, where $q = \pi(\xi)$. If $T \in T_q Q$ is a tangent vector to the base manifold, then its horizontal lift is the unique horizontal vector $\tilde{T} \in H_\xi$ such that $\pi_*\tilde{T} = T$. Given any basis of smooth vector fields in an open neighbourhood of the base manifold, $\{\boldsymbol{f}_m, m = 1, \ldots, \dim Q\}$, their unique horizontal lifts are denoted by $\{\boldsymbol{F}_m, m = 1, \ldots, \dim Q\}$. The set $\{(\boldsymbol{F}_m)_\xi\}$ is a basis for the horizontal subspace H_ξ and

$$(\boldsymbol{F}_m)_\xi = (\boldsymbol{f}_m)_q - \sum_a \Gamma_m^a(\xi)(e_a^*)_S \tag{19}$$

where, in a local trivialization, $\xi = (q; S)$, and the coefficients Γ_m^a are smooth real-valued functions on the bundle $\mathcal{P}$.

The second defining property of a connexion, equation (18), asserts that $(\boldsymbol{F}_m)_{\xi\cdot g} = (R_g)_*(\boldsymbol{F}_m)_\xi$. In particular, when $\xi = (q; I)$, where I is the structure group identity and $g = S^{-1} \in G$, the right translation of a horizontal basis vector at the structure group identity is

$$\begin{aligned}(\boldsymbol{F}_m)_{(q;S)} &= (R_{S^{-1}})_*(\boldsymbol{F}_m)_{(q;I)} \\ &= (\boldsymbol{f}_m)_q - \sum_a \Gamma_i^a(q; I)(Ad_{S^{-1}}e_a)_S^* \\ &= (\boldsymbol{f}_m)_q + \sum_a \Gamma_m^a(q)(\mathcal{R}_{e_a})_S.\end{aligned} \tag{20}$$

The functions $\Gamma_m^a(q) \equiv \Gamma_m^a(q; I)$ are the Christoffel symbols.

Consider now a basis $\{(\boldsymbol{f}_m)_q, m = 1, \ldots, 6\}$ for the tangent space at $q \in Q$ that consists of the three right-invariant vector fields $(\mathcal{R}_{e_i})_R$ on the rotation group $SO(3)$ and the three vibrational vector fields $(\partial/\partial a_i)_A$. A tangent vector to a curve in the base manifold is

$$T(t) = \sum_{i=1}^{3} \big(-\omega_i\, (\mathcal{R}_{e_i})_R + \dot{a}_i\, (\partial/\partial a_i)_A \big). \tag{21}$$

The curve's lift to the bundle is required to have the tangent $V(t)$ of equation (6),

$$V(t) = -\sum_{i=1}^{3} \omega_i \left((\mathcal{R}_{e_i})_R + \frac{\lambda_i}{\omega_i}\, (\mathcal{R}_{e_i})_S \right) + \sum_{i=1}^{3} \dot{a}_i\, (\partial/\partial a_i)_A. \tag{22}$$

The lift is horizontal if and only if H_ξ is spanned by

$$\boldsymbol{F}_i = (\mathcal{R}_{e_i})_R + \left(\frac{\lambda_i}{\omega_i}\right)(\mathcal{R}_{e_i})_S \tag{23}$$

$$\boldsymbol{F}_{i+3} = \frac{\partial}{\partial a_i} \tag{24}$$

for $i = 1, 2, 3$. The Riemann ellipsoid Christoffel symbols vanish for the vibrational vectors and simplify to a diagonal form for the rotational vectors

$$\Gamma_i^a(q) = \delta_i^a \left(\frac{\lambda_i}{\omega_i}\right). \tag{25}$$

In particular, the special rotational modes correspond to the following Christoffel symbols:

$$\Gamma_k^k = \begin{cases} 0 & \text{rigid} \\ 2a_i a_j/(a_i^2 + a_j^2) & \text{irrotational} \\ (a_i^2 + a_j^2)/(2a_i a_j) & \text{falling cat} \end{cases} \tag{26}$$

where i, j, k are cyclic. The Christoffel symbols are just functions of the axis lengths due to rotational invariance of the horizontal subspace, $(L_r)_* H_\xi = H_{r\xi}$.

3.1. Riemannian connexion

The horizontal subspace H_ξ^{IF} for irrotational flow is defined as the orthogonal complement to the vertical subspace V_ξ. Denote the vector space of all 3×3 symmetric matrices by $\mathfrak{m}$. The orthogonal complement $V_\xi^\perp$ is given explicitly by

$$H_\xi^{\mathrm{IF}} = \left\{(\mathcal{R}_Y)_\xi \in T_\xi \mathcal{P} \mid Y \in \mathfrak{m}\right\}. \tag{27}$$

To prove this, suppose $(\mathcal{R}_\Lambda)_S$, $\Lambda \in \mathfrak{g}$, is a vertical vector and $(\mathcal{R}_Y)_\xi$, $Y \in \mathfrak{m}$, is a horizontal vector. These two vectors are orthogonal,

$$\begin{aligned} \boldsymbol{g}_\xi((\mathcal{R}_\Lambda)_S, (\mathcal{R}_Y)_\xi) &= \boldsymbol{g}_\xi((\mathcal{R}_{R^t A \Lambda A^{-1} R})_\xi, (\mathcal{R}_Y)_\xi) \\ &= \operatorname{tr}\left(R^t A \Lambda A^{-1} R(\xi \xi^t) Y^t\right) \\ &= \operatorname{tr}\left(R^t A \Lambda A R Y\right) \\ &= -\operatorname{tr}\left(Y^t R^t A \Lambda A R\right) \\ &= 0. \end{aligned} \tag{28}$$

Since the sums of the dimensions of the vertical space and the horizontal space add up to the dimension of the tangent space $T_\xi \mathcal{P}$, the tangent space is a direct sum of the horizontal and vertical subspaces. If $(\mathcal{R}_Y)_\xi$ is a horizontal vector and $g \in G$, then right invariance implies

$$(R_g)_* (\mathcal{R}_Y)_\xi = (\mathcal{R}_Y)_{\xi g^{-1}} \tag{29}$$

or $(R_g)_* H_\xi^{\mathrm{IF}} = H_{\xi g^{-1}}^{\mathrm{IF}}$. Since the assignment of the horizontal subspace H_ξ^{IF} is also smooth, it defines a connexion on $\mathcal{P}$.

The vibrational vectors are horizontal since $Y = R^t A^{-1} \dot{A} R$ is a symmetric matrix. But the rotational vectors are not horizontal because

$$\boldsymbol{g}_\xi((\mathcal{R}_{e_i})_R, (\mathcal{R}_{e_b})_S) = \operatorname{tr}\left(A e_i A e_b\right) = -2\delta_{ib} a_j a_k \neq 0 \tag{30}$$

for i, j, k cyclic. Note that the inner product of two vertical vectors is also nonzero,

$$\boldsymbol{g}_\xi((\mathcal{R}_{e_a})_S, (\mathcal{R}_{e_b})_S) = -\operatorname{tr}\left(A^2 e_a e_b\right) = \delta_{ab}(a_j^2 + a_k^2) \tag{31}$$

for a, j, k cyclic. In order for $(\boldsymbol{F}_i)_\xi$ to be the horizontal lift of $(\mathcal{R}_{e_i})_R$, it is necessary and sufficient that, for $b = 1, 2, 3$,

$$\begin{aligned} 0 &= \boldsymbol{g}_\xi((\boldsymbol{F}_i)_\xi, (\mathcal{R}_{e_b})_S) \\ &= \boldsymbol{g}_\xi((\mathcal{R}_{e_i})_R + \Gamma_i^a(q)(\mathcal{R}_{e_a})_S, (\mathcal{R}_{e_b})_S) \\ &= -2\delta_{ib}a_j a_k + \Gamma_i^b(q)(a_j^2 + a_k^2). \end{aligned} \tag{32}$$

The off-diagonal Christoffels for the rotational vectors vanish, and the diagonal values are

$$\Gamma_i^i(q) = \frac{2a_j a_k}{(a_j^2 + a_k^2)} \qquad (i, j, k \text{ cyclic}). \tag{33}$$

Thus, the Riemannian connexion for which the horizontal space is perpendicular to the vertical space corresponds to irrotational flow.

3.2. Invariant connexion

The falling cat connexion is the invariant connexion on the Lie group $\mathcal{P}$. Since $\mathfrak{g}$ is the algebra of antisymmetric matrices and $\mathfrak{m}$ is the vector space of symmetric matrices, the Lie algebra of the group $\mathcal{P}$ is a direct sum of vector spaces, $M_3(\mathbb{R}) = \mathfrak{g} \oplus \mathfrak{m}$. Moreover the vector space $\mathfrak{m}$ is invariant with respect to the adjoint group transformation, $Ad_g(\mathfrak{m}) \subset \mathfrak{m}$ for all $g \in G$. These two properties of $\mathfrak{m}$ are necessary and sufficient for

$$H_\xi^{\mathrm{FC}} = \left\{(\mathcal{L}_Y)_\xi = -(\mathcal{R}_{Ad_\xi Y})_\xi \in T_\xi\mathcal{P} \mid Y \in \mathfrak{m}\right\} \tag{34}$$

to be a horizontal subspace [25]. In order to see that, note that the vertical vectors can be expressed in left-invariant form,

$$V_\xi = \left\{(\mathcal{R}_\Lambda)_S = -(\mathcal{L}_{S^t\Lambda S})_\xi \in T_\xi\mathcal{P} \mid \Lambda \in \mathfrak{g}\right\}. \tag{35}$$

The tangent space to the bundle at ξ is a direct sum of the horizontal and vertical subspaces, because every matrix is a linear combination of a symmetric matrix Y and an antisymmetric matrix $S^t\Lambda S$. The right invariance of the horizontal subspaces is a consequence of

$$(R_g)_*(\mathcal{R}_{Ad_\xi Y})_\xi = (\mathcal{R}_{Ad_\xi Y})_{\xi g^{-1}} = (\mathcal{R}_{Ad_{\xi g^{-1}} Ad_g Y})_{\xi g^{-1}} \in H_{\xi g^{-1}}^{\mathrm{FC}} \tag{36}$$

since $Ad_g Y \in \mathfrak{m}$ for all $g \in G$ and $Y \in \mathfrak{m}$. The assignment of the subspaces is smooth, so H_ξ^{FC} is indeed a horizontal subspace.

The relation

$$\dot{a}_i\left(\frac{\partial}{\partial a_i}\right)_A = -(\mathcal{R}_{R^t A^{-1}\dot{A}R})_\xi = (\mathcal{L}_{S^t A^{-1}\dot{A}S})_\xi \tag{37}$$

shows that the vibrational vectors are horizontal ($S^t A^{-1}\dot{A}S$ is symmetric), but the rotational vectors are not horizontal since

$$(\mathcal{R}_{e_i})_R = -(\mathcal{R}_{R^t e_i R})_\xi = (\mathcal{L}_{S^t A^{-1} e_i A S})_\xi \tag{38}$$

and $S^t A^{-1} e_i A S$ is not symmetric. If the matrix $S^t A^{-1} e_i A S$ is expressed as a sum of symmetric X_{s} and antisymmetric X_{a} parts, i.e. $X_{\mathrm{s}} = (S^t A^{-1} e_i A S - S^t A e_i A^{-1} S)/2$, $X_{\mathrm{a}} = (S^t A^{-1} e_i A S + S^t A e_i A^{-1} S)/2$, the angular momentum may be written as a sum of horizontal and vertical vectors:

$$(\mathcal{R}_{e_i})_R = (\mathcal{L}_{X_{\mathrm{s}}})_\xi + (\mathcal{L}_{X_{\mathrm{a}}})_\xi \in H_\xi^{\mathrm{FC}} \oplus V_\xi. \tag{39}$$

The horizontal lifts of the angular momentum vectors are

$$\begin{aligned} (\boldsymbol{F}_i)_\xi &= (\mathcal{R}_{e_i})_R + \Gamma_i^a(q)(\mathcal{R}_{e_a})_S \\ &= (\mathcal{L}_{X_{\mathrm{s}}})_\xi + \left[(\mathcal{L}_{X_{\mathrm{a}}})_\xi - \Gamma_i^a(q)(\mathcal{L}_{S^t e_a S})_\xi\right] \end{aligned} \tag{40}$$

where $(\mathcal{L}_{X_s})_\xi$ is the horizontal lift and the two vertical vectors in the square brackets must cancel. Therefore, the invariant connexion is given by

$$\Gamma_i^a(q)e_a = (A^{-1}e_i A + A e_i A^{-1})/2 \tag{41}$$

or the Christoffel symbols are diagonal and

$$\Gamma_i^i(q) = \frac{(a_j^2 + a_k^2)}{2a_j a_k} \tag{42}$$

where i, j, k are cyclic.

3.3. Dedekind's theorem

The geometrical relationship between the Riemannian and invariant connexions is equivalent to Dedekind's theorem, which relates irrotational flow to falling cat solutions [10]. Define the Dedekind involution $f : \mathcal{P} \to \mathcal{P}$ as the matrix transpose, $f(\xi) = \xi^t$. In a local trivialization, $\xi = R^t A\, S$, the Dedekind map interchanges R and S. It also interchanges the angular velocity Ω and the vortex velocity Λ.

The differential $f_* : T_\xi\mathcal{P} \to T_{\xi^t}\mathcal{P}$ of the Dedekind involution maps a right-invariant vectorfield into a left-invariant one, $f_*(\mathcal{R}_Y)_\xi = -(\mathcal{L}_{Y^t})_{\xi^t}$. Hence f_* is a vector space isomorphism from the irrotational flow horizontal subspace at ξ onto the falling cat horizontal subspace at ξ^t,

$$f_* : H_\xi^{\mathrm{IF}} \longrightarrow H_{\xi^t}^{\mathrm{FC}}. \tag{43}$$

The Riemannian metric is Dedekind invariant,

$$\boldsymbol{g}_{\xi^t}(f_*(\mathcal{R}_X)_\xi, f_*(\mathcal{R}_Y)_\xi) = \boldsymbol{g}_\xi((\mathcal{R}_X)_\xi, (\mathcal{R}_Y)_\xi) \tag{44}$$

because $f_*(\mathcal{R}_X)_\xi = (\mathcal{R}_{Ad_{\xi^t}X^t})_{\xi^t}$. When the potential $V = V(A)$ is a pure function of the axis lengths, then the Lagrangian $L = K - V$ is invariant. Therefore, the Dedekind involution transforms solutions of Lagrange's equations into other solutions, known as the adjoint solutions. If a solution is constrained to irrotational flow, then the Dedekind involution maps it into a falling cat solution, and vice versa.

4. Conclusions

The concept of a horizontal lift is physically natural. It says that a many-body system responds to rotations and vibrations (described by a curve γ in the base manifold) by internal vortex motions (described by a horizontally-lifted curve $\tilde{\gamma}$ in the bundle). This response is determined typically by a nonholonomic constraint that depends ultimately on the nature of the forces between the particles. The constraint that the tangent to the lifted curve lies in a horizontal subspace is equivalent to a bundle connexion.

The connexions corresponding to rigid rotation, irrotational flow, and the falling cat were shown to be natural geometrical or group-theoretical concepts. Although not mathematically natural, other choices of Christoffel symbols define nonholonomic constraint forces that are not excluded by physical law. For example, the S-type Riemann ellipsoids are a sequence of special case solutions for which the angular momentum, Kelvin circulation, and the angular and vortex velocity vectors are aligned with a principal axis, say the 1-axis [9, 10]. This sequence is indexed by a continuous real parameter f restricted to the interval $-2 \leqslant f \leqslant 0$. There is only one horizontal lift to consider and the Christoffel symbol is given by

$$\Gamma_1^1(q) = -\frac{f a_2 a_3}{(a_2^2 + a_3^2)}. \tag{45}$$

At $f = 0$, the connexion yields rigid rotation, and, at $f = -2$, it is irrotational flow. The S-type ellipsoids are the simplest models that allow for a continuous interpolation between rigid rotation and irrotational flow. This connexion has no natural geometrical or group-theoretic significance—but it does model a variety of rotating physical systems.

An unsolved basic science problem is to determine the connexion from the interactions among the particles that form a rotating system. A complete theory of collective rotation requires equations that incorporate these interactions into the gauge theory and whose unique solution are the Christoffel symbols. They must involve a coordinate independent object and the curvature form is the obvious candidate. The Bianchi identity partially determines the Christoffel symbols, but it is not sufficient. There must be another equation that relates the bundle curvature to the microscopic physics.

The stability of equilibrium solutions is a major part of the theory of Riemann ellipsoids [32]. This classical theory might be revisited in the geometrical setting of gauge theory using modern methods [33, 34].

The author would like to thank the Institute of Nuclear Physics and the organizers, J Ginocchio and F Iachello, of the programme entitled, 'Algebraic Methods in Many-Body Physics', for their support. Valuable discussions with J Bryan, Ts Dankova, E Ihrig and J Troupe contributed intellectually to this work.

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